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THE STABILITY OF THE UNSTEADY MOTION OF A MECHANICAL SYSTEM†

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The sufficient conditions for asymptotic stability and instability of the relative equilibrium position of a mechanical system with holonomic-rheonomic constraints are derived. On the basis of this, new methods of solving the problem of stabilizing the programmed motions of the controlled mechanical systems are proposed. The problem of the stability of the equilibrium position and the programmed motion of a physical pendulum, the horizontal swinging axis of which rotates was a variable angular velocity around a vertical axis, is solved. The problem of controlling the relative motions of a centrifuge-type system by a regulated velocity of rotation of the base is investigated.

1. THE STABILITY OF THE RELATIVE EQUILIBRIUM POSITION OF A MECHANICAL SYSTEM WITH HOLONOMIC-RHEONOMIC CONSTRAINTS

Consider a mechanical system with holonomic-rheonomic ideal constraints, the position of which is defined by n generalized coordinates $\mathbf{q}' = (\mathbf{q}_1, \mathbf{q}_2, \dots, \mathbf{q}_n)$, and the kinetic energy of the system is represented in the form

$$T = T_2 + T_1 + T_0$$

$$T_2(t, \mathbf{q}, \dot{\mathbf{q}}) = \frac{1}{2}\dot{\mathbf{q}}'A(t, \mathbf{q})\dot{\mathbf{q}}, \quad T_1(t, \mathbf{q}, \dot{\mathbf{q}}) = B'(t, \mathbf{q})\dot{\mathbf{q}}$$

where $A(t, \mathbf{q})$ is a positive-definite $n \times n$ matrix, $B(t, \mathbf{q})$ is and $n \times 1$ column matrix and $T_0(t, \mathbf{q})$ is a scalar function; the prime denotes transposition.

The motion of the system under potential forces with a potential energy $\Pi(t, \mathbf{q})$ and other generalized forces $\mathbf{Q} = \mathbf{Q}(t, \mathbf{q}, \dot{\mathbf{q}})$ can be described by Lagrange's equations, reduced to the form

$$\frac{d}{dt}\left(\frac{\partial T_2}{\partial \dot{\mathbf{q}}}\right) - \frac{\partial T_2}{\partial \mathbf{q}} = -\frac{\partial W}{\partial \mathbf{q}} - G\dot{\mathbf{q}} - \frac{\partial B}{\partial t} + \mathbf{Q} \tag{1.1}$$

The matrix G is defined by the equality

$$G(t, \mathbf{q}) = \frac{\partial B}{\partial \mathbf{q}} - \left(\frac{\partial B}{\partial \mathbf{q}}\right)' = -G'$$

and can be regarded as the matrix of linear gyroscopic forces, and $W(t, \mathbf{q}) = \Pi(t, \mathbf{q}) - T_0(t, \mathbf{q})$ can be defined as the reduced potential energy.

We will assume that, for a certain value of $\mathbf{q} = \mathbf{q}_0$, for all $t \in R^+$ we have the equality

$$\mathbf{Q}(t, \mathbf{q}_0, 0) - \frac{\partial W}{\partial \mathbf{q}}(t, \mathbf{q}_0) - \frac{\partial B}{\partial t}(t, \mathbf{q}_0) \equiv 0 \tag{1.2}$$

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System (1.1) then has the relative equilibrium position

$$\dot{\mathbf{q}}(t) = 0, \quad \mathbf{q}(t) = \mathbf{q}_0 \tag{1.3}$$

We will consider the problem of its stability, assuming that the elements of the matrices $A(t, \mathbf{q})$ and $B(t, \mathbf{q})$ and the function $W(t, \mathbf{q})$ are defined and twice continuously differentiable in the region $R^+ \times \Gamma_1$, and the vector function $\mathbf{Q}(t, \mathbf{q}, \dot{\mathbf{q}})$ is defined and continuously differentiable in the region $R^+ \times \Gamma_1 \times \Gamma_2$, where all the functions indicated are uniformly bounded together with their derivatives for bounded $||\mathbf{q}||$ and $||\dot{\mathbf{q}}||$ for all $t \in R^+$. Here

$$\Gamma_i = \{\mathbf{q} \in R^n : \|\mathbf{q}\| < H_i, 0 < H_i \le +\infty\}, \quad i = 1, 2$$

and $||\mathbf{q}||$ is the Euclidean norm of the vector $\mathbf{q} \in R^n$, $||\mathbf{q}||^2 = \mathbf{q}_1^2 + \ldots + \mathbf{q}_n^2$.

It follows from these conditions, imposed on the functions in Eqs (1.1), that Eqs (1.1) are precompact [1, 2], and limit equations are defined for these which have a form similar to Eqs (1.1),

$$\frac{d}{dt}\left(\frac{\partial T_2^*}{\partial \dot{\mathbf{q}}}\right) - \frac{\partial T_2^*}{\partial \mathbf{q}} = -\frac{\partial W^*}{\partial \mathbf{q}} - G^*\dot{\mathbf{q}} - \frac{\partial B^*}{\partial t} + \mathbf{Q}^* \tag{1.4}$$

The asterisks denote functions, matrices and expressions, which are limit functions, matrices and expressions for the corresponding functions, matrices and expressions of Eqs (1.1) and are defined by the following equations (the limit is taken as $t_n \to +\infty$)

$$T_2^*(t, \mathbf{q}, \dot{\mathbf{q}}) = \frac{1}{2}\dot{\mathbf{q}}' A^*(t, \mathbf{q})\dot{\mathbf{q}}, \quad A^*(t, \mathbf{q}) = \lim A(t_n + t, \mathbf{q})$$

$$W^*(t, \mathbf{q}) = \lim W(t_n + t, \mathbf{q}), \quad \mathbf{Q}^*(t, \mathbf{q}, \dot{\mathbf{q}}) = \lim \mathbf{Q}(t_n + t, \mathbf{q}, \dot{\mathbf{q}})$$

$$G^*(t, \mathbf{q}) = \lim G(t_n + t, \mathbf{q}), \quad B^*(t, \mathbf{q}) = \lim B(t_n + t, \mathbf{q})$$

Here the corresponding convergence is uniform with respect to

$$(t, \mathbf{q}, \dot{\mathbf{q}}) \in [0, T] \times \{\mathbf{q} : \|\mathbf{q}\| \le H_0 < H_1\} \times \{\dot{\mathbf{q}} : \|\dot{\mathbf{q}}\| \le H_0 < H_1\}$$

The limit equations (1.4) define limit properties of the motions of system (1.1), and this enables us, according to theorems proved previously [1], to obtain the sufficient conditions for asymptotic stability and instability of the unperturbed motion using Lyapunov functions, which have a sign-constant derivative.

For convenience we will denote the Hahn-type function by h [3], $h : R^+ \times R^+$, $h(0) = 0$ and $h(a)$ is a strictly monotonically increasing continuous function; we will denote by $\gamma : R^+ \to R^+$ the uniformly continuous function, positive on average, i.e. such that for a certain $T > 0$

$$\int_t^{t+T} \gamma(\tau) d\tau \ge \gamma_0 > 0, \quad \forall t \in R^+$$

Consider the region

$$D = R^+ \times \{(\mathbf{q}, \dot{\mathbf{q}}) : \|\mathbf{q} - \mathbf{q}_0\| < \delta, \|\dot{\mathbf{q}}\| < \delta\}$$

The following assertions regarding the stability of the equilibrium position (1.3) of system (1.1) hold.

Assertion 1.1. We will assume that Eq. (1.2) holds and

(1) the function $W(t, \mathbf{q}) - W_0(t)$, $W_0(t) = W(t, \mathbf{q}_0)$ in the neighbourhood $\{\mathbf{q} : ||\mathbf{q} - \mathbf{q}_0|| < \delta > 0\}$ of the point $\mathbf{q} = \mathbf{q}_0$ is positive-definite and allows of an infinitesimal higher limit with respect to $\mathbf{q} - \mathbf{q}_0$

$$h_1(\|\mathbf{q} - \mathbf{q}_0\|) \le W(t, \mathbf{q}) - W_0(t) \le h_2(\|\mathbf{q} - \mathbf{q}_0\|) \tag{1.5}$$

(2) the acting forces and constraints are such that

$$\frac{\partial}{\partial t}(-T_2 - T_1 + W - W_0) + \dot{\mathbf{q}}'\mathbf{Q} \le 0, \quad \forall (t, \mathbf{q}, \dot{\mathbf{q}}) \in D$$

The equilibrium position (1.3) is then uniformly stable.

Assertion 1.2. We will assume that Eq. (1.2) holds and
(1) for all $\mathbf{q} \in \{\mathbf{q} : \|\mathbf{q}-\mathbf{q}_0\| < \delta > 0\}$ conditions (1.5) are satisfied;
(2) the acting forces and constraints are such that

$$\frac{\partial}{\partial t}(-T_2 - T_1 + W - W_0) + \dot{\mathbf{q}}'\mathbf{Q} \le -\gamma(t)h_3(\|\dot{\mathbf{q}}\|) \le 0, \quad \forall(t, \mathbf{q}, \dot{\mathbf{q}}) \in D$$

(3) the relative equilibrium position (1.3) is isolated: for any $\eta > 0$ we obtain $\varepsilon = \varepsilon(\eta) > 0$ such that when $t \ge t_0$

$$\left\|\mathbf{Q}(t, \mathbf{q}, 0) - \frac{\partial W(t, \mathbf{q})}{\partial \mathbf{q}} - \frac{\partial B(t, \mathbf{q})}{\partial t}\right\| \ge \varepsilon, \quad \forall \mathbf{q} \in \{0 < \eta \le \|\mathbf{q}-\mathbf{q}_0\| \le \delta\}$$

Equilibrium position (1.3) is then uniformly asymptotically stable.

Assertions 1.1 and 1.2 are derived from the theorem of stability in [3, 4] and from the theorem on asymptotic stability in [1] using the function $V = T_2 + W - W_0$.

Assertion 1.3. We will assume that equality (1.2) holds, there is no gyroscopic component of the inertial forces ($G \equiv 0$), and also in the neighbourhood $\{\mathbf{q} : \|\mathbf{q}-\mathbf{q}_0\| < \delta > 0\}$ of the point $\mathbf{q} = \mathbf{q}_0$ for all $t \ge t_0$ the following inequality holds

$$(\mathbf{q}-\mathbf{q}_0)'\left(\mathbf{Q}(t, \mathbf{q}, \dot{\mathbf{q}}) - \frac{\partial W(t, \mathbf{q})}{\partial \mathbf{q}} - \frac{\partial B(t, \mathbf{q})}{\partial t}\right) \ge 0 \tag{1.6}$$

The equilibrium position (1.3) of system (1.1) is unstable.

Assertion 1.4. The conclusion regarding the instability of equilibrium position (1.3) remains true if, instead of condition (1.6), we assume that the forces $\mathbf{Q}$ are linear dissipative forces $\mathbf{Q} = -R\dot{\mathbf{q}}$, $R = \text{const}$, and also

$$(\mathbf{q}-\mathbf{q}_0)'\left(\frac{\partial W(t, \mathbf{q})}{\partial \mathbf{q}} + \frac{\partial B(t, \mathbf{q})}{\partial t}\right) \le 0$$

Assertions 1.3 and 1.4 are derived from the theorem of instability in [1] using respectively the Lyapunov functions

$$V_1(t, \mathbf{q}, \dot{\mathbf{q}}) = -(\mathbf{q}-\mathbf{q}_0)'\frac{\partial T_2(t, \mathbf{q}, \dot{\mathbf{q}})}{\partial \dot{\mathbf{q}}}$$

$$V_2(t, \mathbf{q}, \dot{\mathbf{q}}) = -(\mathbf{q}-\mathbf{q}_0)'\frac{\partial T_2(t, \mathbf{q}, \dot{\mathbf{q}})}{\partial \dot{\mathbf{q}}} - \frac{1}{2}(\mathbf{q}-\mathbf{q}_0)'R(\mathbf{q}-\mathbf{q}_0)$$

We will assume that the constraints and acting forces are such that the following representation holds

$$\begin{gathered}\mathbf{Q} = \mathbf{Q}_1 + \mathbf{Q}_2 \\ \mathbf{Q}_1(t, \mathbf{q}) - \frac{\partial W}{\partial \mathbf{q}}(t, \mathbf{q}) - \frac{\partial B}{\partial t}(t, \mathbf{q}) = -p(t, \mathbf{q})\frac{\partial S}{\partial \mathbf{q}}(\mathbf{q}) \\ \mathbf{Q}_2 = \mathbf{Q}_2(t, \mathbf{q}, \dot{\mathbf{q}}), \quad \mathbf{Q}_2(t, \mathbf{q}, 0) \equiv 0\end{gathered} \tag{1.7}$$

where $p(t, \mathbf{q})$ and $S(\mathbf{q})$ are scalar functions, twice continuously differentiable with respect to $t \in R^+$ and $\mathbf{q} \in \Gamma_1$, where $0 < p_0 \le p(t, \mathbf{q}) \le p_1$.

Assertion 1.5. Suppose the forces can be represented in the form (1.7), and also
(1) for a certain value of $\mathbf{q} = \mathbf{q}_0$

$$\frac{\partial S(\mathbf{q}_0)}{\partial \mathbf{q}} = 0, \quad S(\mathbf{q}) - S(\mathbf{q}_0) \ge h_1(\|\mathbf{q}-\mathbf{q}_0\|)$$

(2) the corresponding relative equilibrium position (1.3) of system (1.1) is isolated, i.e.

$$\left\|\frac{\partial S}{\partial \mathbf{q}}\right\| \neq 0, \quad \mathbf{q} \in \{0 < \|\mathbf{q}-\mathbf{q}_0\| < \delta\} \tag{1.8}$$

(3) the following relation holds

$$-\frac{1}{p^2}\left(\frac{\partial p}{\partial l} + \dot{\mathbf{q}}'\frac{\partial p}{\partial \mathbf{q}}\right)T_2 - \frac{1}{p}\frac{\partial T_2}{\partial t} + \frac{1}{p}\dot{\mathbf{q}}'\mathbf{Q}_2 \leq -\gamma(t)h_2(\|\dot{\mathbf{q}}\|) \leq 0, \ (t, \mathbf{q}, \dot{\mathbf{q}}) \in D$$

Then equilibrium position (1.3) is uniformly asymptotically stable.

We will assume that the constraints imposed on the system and the acting forces are such that

$$\mathbf{Q}_1(t, \mathbf{q}) - \frac{\partial W}{\partial \mathbf{q}}(t, \mathbf{q}) - \frac{\partial B}{\partial t}(t, \mathbf{q}) = -P(t, \mathbf{q})\frac{\partial S(\mathbf{q})}{\partial \mathbf{q}} \tag{1.9}$$

where $S(\mathbf{q})$ is the function introduced above, $P(t, \mathbf{q})$ is an $n \times n$ matrix, twice continuously differentiable, bounded, and non-degenerate, $|\det P| \geq \alpha_0 = \text{const} > 0$, in which case the matrix $P^{-1}(t, \mathbf{q})A(t, \mathbf{q})$ is positive-definite, and $P^{-1}(t, \mathbf{q})A(t, \mathbf{q}) \geq \gamma_0 E$ or $(t, \mathbf{q}) \in R^+ \times \Gamma_1$.

Assertion 1.6. We will assume that (1.9) and conditions 1 and 2 of Assertion (1.5) are satisfied, and also that the following relations holds

$$\dot{\mathbf{q}}'P^{-1}\left(G'\dot{\mathbf{q}} + \mathbf{Q}_2 + \frac{\partial T_2}{\partial \mathbf{q}}\right) - \frac{1}{2}\dot{\mathbf{q}}'P^{-1}\frac{dP}{dt}P^{-1}A\dot{\mathbf{q}} - \frac{1}{2}\dot{\mathbf{q}}'P^{-1}\frac{dA}{dt}\dot{\mathbf{q}} \leq -\gamma(t)h_2(\|\dot{\mathbf{q}}\|) \leq 0 \ (t, \mathbf{q}, \dot{\mathbf{q}}) \in D$$

Equilibrium position (1.3) is then uniformly asymptotically stable.

Assertions (1.5) and (1.6) were derived from the theorem of asymptotic stability in [1] using the following functions, respectively.

$$V_3 = \frac{T_2(t, \mathbf{q}, \dot{\mathbf{q}})}{p(t, \mathbf{q})} + S(\mathbf{q}), \quad V_4 = \frac{1}{2}\dot{\mathbf{q}}'P^{-1}(t, \mathbf{q})A(t, \mathbf{q})\dot{\mathbf{q}} + S(\mathbf{q})$$

Example 1.1. Consider a physical pendulum [5], the horizontal swinging axis OO' of which rotates around the vertical axis ON in accordance with the transient relation $\omega = \omega(t)$. Suppose the lines OO' and OG, where the point G is the centre of gravity of the body, are the principal axes of the ellipsoid of inertia of the body for the point O, $z_0 = |OG|$.

We will introduce a rectangular system of coordinates $Oxyz$, connected with the body, directing the x and z axes along OO' and OG respectively, while the y axis is orthogonal to the x and z axes. We will take ϑ – the angle between the downward vertical and the z axis (Fig. 1) – as the generalized coordinate.

Suppose A, B and C are the moments of inertia of the body about the x, y, z axes.

We will find the components of the kinetic energy and the reduced potential energy

$$T_2 = \frac{1}{2}A\dot{\vartheta}^2, \quad T_1 = 0, \quad W = -mgz_0\cos\vartheta - \frac{1}{2}\omega^2(t)(B\sin^2\vartheta + C\cos^2\vartheta)$$

The relative equilibrium positions are found from the equation

$$\frac{\partial W}{\partial \vartheta} = (\omega^2(t)(C-B)\cos\vartheta + mgz_0)\sin\vartheta = 0$$

which, for any $\omega(t)$, has the solutions

$$\vartheta = 0, \quad \dot{\vartheta} = 0 \tag{1.10}$$

$$\vartheta = \pi, \quad \dot{\vartheta} = 0 \tag{1.11}$$

From Assertion 1.1 we obtain the following conditions for uniform stability of the relative equilibrium positions

$$\pm mgz_0 + (C-B)\omega^2(t) \geq \alpha_0 > 0, \ (C-B)\omega(t)\dot{\omega}(t) \leq 0 \tag{1.12}$$

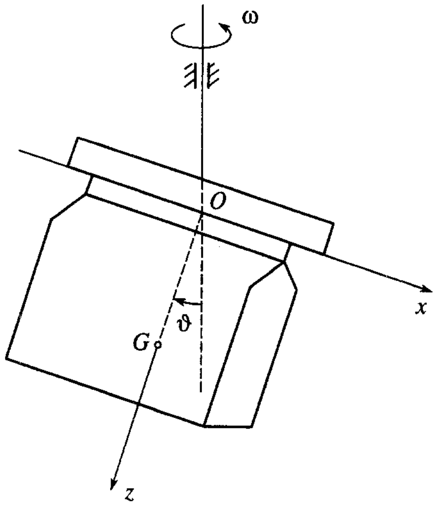

Fig. 1

where the upper sign is taken for relative equilibrium position (1.10) and the lower sign for relative equilibrium position (1.11).

We will assume that, in addition to gravity, viscous friction forces also act on the body. These produce a moment $M_\vartheta = -k\dot{\vartheta}$, k = const > 0. Then, by Assertion 1.2 we have that relative equilibrium positions (1.10) and (1.11) with conditions (1.12) are uniformly asymptotically stable.

We will introduce the functions

$$p(t, \vartheta) = \pm mgz_0 + (C-B)\omega^2(t)\cos\vartheta, \quad S(\vartheta) = 1 \mp \cos\vartheta$$

(as above, the upper sign is taken for relative equilibrium position (1.10) and the lower sign for relative equilibrium condition position (1.11)).

Using Assertion (1.5) we can obtain the following conditions for uniform asymptotic stability of the relative equilibrium positions

$$p(t, 0) \geq \alpha_0 > 0, \quad k(t) \geq k_0 - \frac{A(C-B)\omega(t)\dot{\omega}(t)}{p(t, 0)}$$

From Assertion 1.3 we can obtain that, when there are viscous friction forces and when there are no such forces, the conditions

$$p(t, 0) \leq -\alpha_0 < 0$$

will be the sufficient conditions for instability of relative equilibrium positions (1.10) and (1.11).

2. STABILIZATION OF THE PROGRAMMED MOTION OF A MECHANICAL SYSTEM

Suppose the position of a controlled holonomic mechanical system is defined by n generalized coordinates $q_1, q_2, \ldots, q_n$, and its motion due to the action of a set of controlling and external forces $\mathbf{Q} = \mathbf{Q}_y + \mathbf{Q}_b$ is described by Lagrange's equations of the second kind.

Suppose $(\mathbf{q}^0(t), \dot{\mathbf{q}}^0(t))$ is the programmed motions of the system, which is produced by controlling forces $\mathbf{Q}_y^0 = \mathbf{Q}_y^0(t)$. We consider the problem of stabilizing this motion, which consists of determining the stabilizing forces $\mathbf{Q}_c$, $\mathbf{Q}_y = \mathbf{Q}_y^0(t) + \mathbf{Q}_c$, which will ensure asymptotic stability of this programmed motion [6].

If we introduce new generalized coordinates $\mathbf{x} = \mathbf{q} - \mathbf{q}^0(t)$, the problem reduces to the problem of determining the stabilizing forces $\mathbf{Q}_c = \mathbf{Q}_c(t, \mathbf{x}, \dot{\mathbf{x}})$, which, when there is simultaneous action of the external forces $\mathbf{Q}_b = \mathbf{Q}_b(t, \mathbf{x}, \dot{\mathbf{x}})$, ensures asymptotic stability of the zeroth equilibrium position $\dot{\mathbf{x}} = \mathbf{x} = 0$ of the system with kinetic energy

$$T = T_2(t, \mathbf{x}, \dot{\mathbf{x}}) + T_1(t, \mathbf{x}, \dot{\mathbf{x}}) + T_0(t, \mathbf{x})$$

Without loss of generality we can assume that the generalized coordinates $\mathbf{q}$ are chosen in such a way that the programmed motion of the system is the equilibrium position $\dot{\mathbf{q}} \equiv 0$, $\mathbf{q} \equiv 0$, and correspondingly the motion of the system is described by equations similar to Eqs (1.1)

$$\frac{d}{dt}\left(\frac{\partial T_2}{\partial \dot{\mathbf{q}}}\right) - \frac{\partial T_2}{\partial \mathbf{q}} = \frac{\partial T_0}{\partial \mathbf{q}} - \frac{\partial B}{\partial t} - G\dot{\mathbf{q}} + \mathbf{Q}_c + \mathbf{Q}_b \tag{2.1}$$

where $G' = -G$ and $\mathbf{Q}_c$ and $\mathbf{Q}_b$ are the stabilizing (controlling) and external (natural) forces respectively. For the programmed motion $\dot{\mathbf{q}} = \mathbf{q} = 0$ to exist, the acting forces must satisfy a relation similar to (1.2)

$$\mathbf{Q}_c(t, 0, 0) + \mathbf{Q}_b(t, 0, 0) + \frac{\partial T_0}{\partial \mathbf{q}}(t, 0) - \frac{\partial B}{\partial t}(t, 0) \equiv 0$$

The problem of stabilizing the programmed motion can be solved starting from assertions similar to Assertion 1.2, 1.5 and 1.6.

We will assume that the forces $\mathbf{Q}_c$ and $\mathbf{Q}_b$ can be separated:

$$\mathbf{Q}_c(t, \mathbf{q}, \dot{\mathbf{q}}) = \mathbf{Q}_c^1(t, \mathbf{q}) + \mathbf{Q}_c^2(t, \mathbf{q}, \dot{\mathbf{q}}), \quad \mathbf{Q}_b = \mathbf{Q}_b^1(t, \mathbf{q}) + \mathbf{Q}_b^2(t, \mathbf{q}, \dot{\mathbf{q}})$$

$$\mathbf{Q}_c^2(t, \mathbf{q}, 0) \equiv 0, \quad \mathbf{Q}_b^1(t, \mathbf{q}, 0) \equiv 0$$

while the components of the stabilizing forces $\mathbf{Q}_c^1$ are chosen in such a way that the following relations are satisfied

$$\mathbf{Q}_c^1(t, \mathbf{q}) + \mathbf{Q}_b^1(t, \mathbf{q}) - \frac{\partial B(t, \mathbf{q})}{\partial t} + \frac{\partial T_0(t, \mathbf{q})}{\partial \mathbf{q}} = -p(t, \mathbf{q})\frac{\partial S(\mathbf{q})}{\partial \mathbf{q}}$$

where the functions $p(t, \mathbf{q})$ and $S(\mathbf{q})$ satisfy condition (1.8) and assumptions (1) and (2) of Assertion (1.5) when $\mathbf{q}_0 = 0$, and the components of the stabilizing forces $\mathbf{Q}^2{}_c$ are chosen in such a way that for all $(t, \mathbf{q}, \dot{\mathbf{q}}) \in D$

$$-\frac{1}{p^2}\left(\frac{\partial p}{\partial t} + \dot{\mathbf{q}}'\frac{\partial p}{\partial \mathbf{q}}\right)T_2 - \frac{1}{p}\frac{\partial T_2}{\partial t} + \frac{1}{p}\dot{\mathbf{q}}'(\mathbf{Q}_c^2 + \mathbf{Q}_b^2) < -\gamma(t)h_2(\|\dot{\mathbf{q}}\|) \tag{2.2}$$

Then, by Assertion 1.5, the stabilizing force $\mathbf{Q}_c$ ensure uniform asymptotic stability of the programmed motion $\dot{\mathbf{q}} = \mathbf{q} = 0$.

Remark 1. If the stabilizing forces $\mathbf{Q}_c = \mathbf{Q}_c^1 + \mathbf{Q}_c^2$ are defined in such a way that conditions (1.7), (1.8) and (2.2) are satisfied for all $(t, \mathbf{q}, \dot{\mathbf{q}}) \in R^+ \times \Gamma_1 \times \Gamma_2$, where $S(\mathbf{q}) \to +\infty$ as $\mathbf{q} \to \partial\Gamma_1$, the force $\mathbf{Q}_c$ will ensure uniform asymptotic stability of the equilibrium position $\dot{\mathbf{q}} = \mathbf{q} = 0$ as a whole, i.e. with respect to any initial perturbations $(t_0, \mathbf{q}_0, \dot{\mathbf{q}}) \in R^+ \times \Gamma_1 \times \Gamma_2$.

Remark 2. The problem of stabilizing programmed motion was also considered earlier, for example, it was solved for controlling forces of the form [7, 8]

$$\mathbf{Q}_y = \dot{B} + \frac{1}{2}\dot{A}\dot{\mathbf{q}} - \frac{\partial T_0}{\partial \mathbf{q}} - \mathbf{Q}_b - A_1\mathbf{q} - B_1\dot{\mathbf{q}}$$

$$\mathbf{Q}_y = \dot{B} + \frac{1}{2}\dot{A}\dot{\mathbf{q}} - \frac{\partial T_0}{\partial \mathbf{q}} - \mathbf{Q}_b - A_1\mathbf{q} - B_1\dot{\mathbf{q}} - gF(\dot{\mathbf{q}} + fC_1A\mathbf{q})$$

where f and g are non-negative numbers, A_1, B_1 and C_1 are constant symmetrical $n \times n$ matrices, which satisfy estimates with positive constants a_i, b_i and c_i $(i = 1, 2)$ respectively:

$$a_1E_n \le A_1 \le a_2E_n, \quad b_1E_n \le B_1 \le b_2E_n, \quad c_1E_n \le C_1 \le c_2E_n$$

and F is a constant matrix, which satisfies the estimate

$$d_1 E_n \le (F + F')/2 \ (d_1 > 0)$$

With the representation, these forces depend very much on the parameters of the mechanical system and do not take into account the action of the external forces, which may turn out to have a stabilizing effect.

Example 2.1. For a physical pendulum (see Example 1.1 in the case of viscous friction forces) we will consider the problem of stabilizing a certain programmed unsteady motion of the pendulum $\vartheta = \vartheta_0(t)$, which is produced by a regulated velocity of rotation $\omega(t)$ around the vertical axis ON. Suppose the rotation is given by $\omega = \omega(t)$ such that the pendulum moves in an unsteady manner: $\vartheta = \vartheta_0(t)$, i.e.

$$\omega^2(t)(C-B)\cos\vartheta_0(t)\sin\vartheta_0(t) = A\ddot{\vartheta}_0(t) + mgz_0 - k\dot{\vartheta}_0(t)$$

If we introduce $x = \vartheta - \vartheta_0(t)$ – the deviation of the actual motion from the programmed motion, the equations of perturbed motion can be written in the form

$$\ddot{x} = -p(t, x)\frac{\partial S}{\partial x} - k_0\dot{x}$$

$$p(t, x) = \frac{1}{A}\left(mgz_0\cos\left(\vartheta_0(t) + \frac{x}{2}\right) + (C-B)\omega^2(t)\cos 2\left(\vartheta_0(t) + \frac{x}{2}\right)\cos\frac{x}{2}\right)$$

$$S(x) = 4\left(1 - \cos\frac{x}{2}\right), \quad k_0 = \frac{k}{A}$$

When the conditions

$$p(t, 0) = mgz_0\cos\vartheta_0(t) + (C-B)\omega^2(t)\cos 2\vartheta_0(t) \ge p_0 > 0$$

$$\frac{d}{dt}(\ln p(t, 0)) \ge -2k_0 + \alpha_0, \quad \alpha_0 = \text{const} > 0$$

are satisfied, the specified motion of the pendulum $\vartheta = \vartheta_0(t)$ is asymptotically stable. These conditions can be represented as the condition for the second variation of the reduced potential energy W to be positive definite in the motion $\vartheta = \vartheta_0(t)$, and the condition for the logarithmic change of this variation with time to have a lower limit of $-2k_0$.

Example 2.2. We will consider the problem of the stability of programmed unsteady motions of a centrifuge (Fig. 2).

The cage of the centrifuge is a rigid body which can rotate freely around an axis OO' about the holder COO'. The axis OO' is orthogonal to the plane L, passing through the axis CC' of the centrifuge and the centre of mass G of the cage. The holder COO' is set in rotation around a fixed axis CC', and the rotation velocity varies as $\omega = \omega(t)$. We take as the generalized coordinate the angle of rotation α of the cage around the axis OO'.

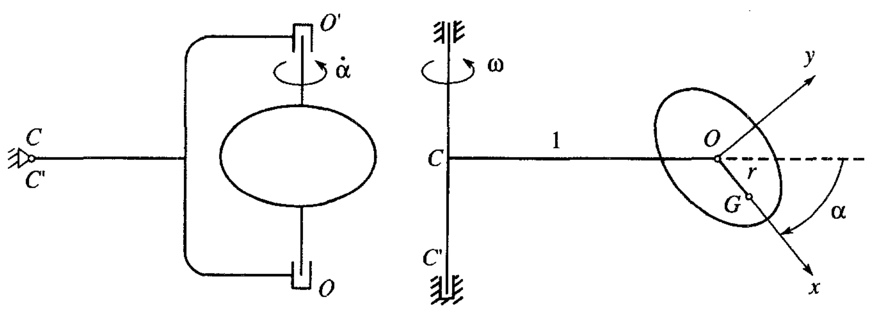

Fig. 2

Suppose L is the plane of symmetry of the cage, while the x axis, passing through the point G in this plane and intersecting the axis OO' at the point O, is the principal axis of the central ellipsoid of inertia. We will denote by A, B and C the moments of inertia of the cage about the x, y, z axes, the distance OC by 1, and the distance OG by r [5].

Correspondingly we obtain

$$T = \frac{1}{2}C\dot{\alpha}^2 + \frac{1}{2}\omega^2(t)(A\sin^2\alpha + B\cos^2\alpha + 2mlr\cos\alpha + ml^2)$$

$$\Pi = mgr\sin\alpha$$

We will assume that forces of viscous friction act in the hinged supports OO', producing a moment

$$M = -k\dot{\alpha}, \quad k = \text{const} > 0 \tag{2.3}$$

Suppose the regulated rotation of the holder COO' around the axis CC' as given by $\omega = \omega(t)$ is such that the cage moves in an unsteady manner: $\alpha = \alpha_0(t)$, i.e.

$$\omega^2(t)((A-B)\cos\alpha_0(t) - mlr)\sin\alpha_0(t) = C\ddot{\alpha}_0(t) + k\dot{\alpha}_0(t) + mgr\cos\alpha_0(t)$$

From Assertion 1.5 we can obtain that for all conditions

$$p(t) = -mgr\sin\alpha_0(t) + \omega^2(t)(mlr\cos\alpha_0(t) - (A-B)\cos 2\alpha_0(t)) \geq p_0 > 0$$

$$\frac{d}{dt}(\ln p(t)) \geq -2k_0 + \beta_0$$

the motion $\alpha = \alpha_0(t)$ will be uniformly asymptotically stable. When $p(t) \leq -p_0 < 0$ it will be unstable.

Example 2.3. Consider the problem of Example 2.2 assuming that the axis OO' is parallel to the axis CC' of the centrifuge. As before, we will take as the generalized coordinate the angle of rotation α of the cage around the axis OO'. We will denote the moment of inertia of the cage about the axis OO' by I [5].

We will assume that forces of viscous friction act in the hinged supports OO', producing a moment (2.3). We will write the equations of motion in the form

$$I\ddot{\alpha} = -\omega^2(t)mlr\sin\alpha + u(2mlr\omega(t)\cos\alpha + I) - k\dot{\alpha}, \quad u = \dot{\omega}(t)$$

Suppose the regulated rotation of the holder COO' around the axis CC' $\omega = \omega(t)$ due to the action of the controlling moment u is such that the cage moves in an unsteady manner: $\alpha = \alpha_0(t)$.

The equation of perturbed motion can be reduced to the form

$$\ddot{x} = -p(t, x)\frac{\partial S}{\partial x} - k_0\dot{x}, \quad x = \alpha - \alpha_0(t)$$

$$p(t, x) = \frac{1}{I}mlr\omega(t)\left(\omega(t)\cos\left(\alpha_0(t) + \frac{x}{2}\right) - \dot{\omega}(t)\sin\left(\alpha_0(t) + \frac{x}{2}\right)\right)$$

$$S(x) = 4\left(1 - \cos\frac{x}{2}\right), \quad k_0 = \frac{k}{I}$$

From Assertion 1.5, when the following condition is satisfied

$$p(t, 0) = \omega(t)(\omega(t)\cos\alpha_0(t) - u(t)\sin\alpha_0(t)) \geq p_0 = \text{const} > 0$$

$$\frac{d}{dt}(\ln p(t, 0)) \geq -2k_0 + \beta_0$$

we will have uniform asymptotic stability of the specified unsteady motion $\alpha = \alpha_0(t)$.

With the opposite condition $p(t, 0) \leq -p_0 < 0$, according to Assertion 1.4, the motion $\alpha = \alpha_0(t)$ will be unstable.

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